The Use of Symbols in Mathematics and Logic

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Abstract. It is commonly believed that the use of arbitrary symbols and the process of symbolisation have made possible the discourse of modern mathematics as well as modern, symbolic logic. This paper discusses the role of symbols in logic and mathematics, and in particular analyses whether symbols remain arbitrary in the process of symbolisation. It begins with a brief summary of the relation between sign and logic as exemplified in Indian logic in order to illustrate a logical system where the notion of 'natural' sign-signified relation is privileged. Mathematics uses symbols in creative ways. Two such methods, one dealing with the process of 'alphabetisation' and the other based on the notion of 'formal similarity', are described. Through these processes, originally meaningless symbols get embodied and coded with meaning through mathematical writing and praxis. It is also argued that mathematics and logic differ in the way they use symbols. As a consequence, logicism becomes untenable even at the discursive level, in the ways in which symbols are created, used and gather meaning.

The role of symbols in the formation of the disciplines of logic (particularly modern and symbolic logic) and mathematics is often acknowledged to be of fundamental importance. However, symbols have become so essential that their function in these disciplines is rarely queried. In the epoch of any discipline it is always worthwhile to periodically reconsider the foundational elements. It is in this spirit that I approach the reconsideration of the role of symbols in logic and mathematics.

Signs, in the most fundamental sense of the word, can refer to anything which stands for something else (the signified). Thus, a word is a sign; for example, the word 'cow' stands for the object cow. There are many ways by which a sign can come to stand for something else. There could be a natural relation which immediately suggests the relation between a sign and the signified. Or, the relation could be arbitrary and chosen through some convention such as words in a language. In what follows, I will sometimes use 'sign' and 'symbol' interchangeably, although I will clarify, where needed, the specific meanings associated with these terms.

What constitutes the process of symbolisation? Firstly, it is the replacement of something by a symbol. There are different kinds of such replacement. For example, one can replace 'mass' by m, a number by 'n', a particular number 'nine' as 9, the idea of 'variable' by x, the concept of relation/mapping by f (as in function) and so on. One can also 'name' objects by a symbol such as using 2 to name the 'object two'. In almost all cases such replacement or naming is conventional and arbitrary. A common example in logic is to symbolise terms such as replacing the sentence 'All Greeks are mortals' by 'All A is B', where A stands for the set of all Greeks and B for all mortals. The conventional and meaninglessness of the symbolisation lies in the fact that A is chosen arbitrarily to stand for all Greeks. Note that neither is there any connection between Greeks and A nor that in performing this replacement any new meaning/information to the original term has been added. The process of symbolisation should not and does not modify or distort that which it stands for. Symbolisation in logic and mathematics most often rests on such premises.

One consequence of this arbitrariness is the concomitant belief that being arbitrary is also to be meaningless, that symbols do not in themselves carry or communicate meaning. This character has often been interpreted as a 'strength' of symbolisation in logic and mathematics. We will later see how logical formulations can be involved in the notion of necessary, non-arbitrary connections even at the level of 'symbols', a classic example being the system of logic that is often referred to as Indian logic. Interestingly, we also find, especially in applied mathematics, similar non-arbitrary, necessary relations between symbols and what they stand for.

Reflecting on the act of symbolisation also illustrates how mathematics and logic can be distinguished. There has been much discussion on whether mathematics and logic are distinguishable, whether mathematics is completely reducible to logic and so on, issues which are well discussed in logicism. However, logicism relates mathematics and logic at the level of content and therefore equality of meaning, as well as the possibility of reducing mathematics to logic. In doing this, the important and differing roles of symbols in both these disciplines are overlooked. There are at least two significant differences in how symbols are used in logic and mathematics. One is the domain of meaning that comes to gather around symbols and the other is the creative ways by which symbols are created and written in mathematics, moves which betray a more fertile relationship between mathematics and the symbolisation process in contrast to logic.

I will first begin this discussion with a brief summary of Indian logic, mainly with respect to its relation with signs and the demands it places on a sign. Since the arbitrary nature of the (valid) sign is what is at stake in Indian logic, I will discuss the nature of arbitrary symbolisation that was seen to be so crucial to

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the formation of logic and mathematics in the Western tradition. I will then look at the use of symbols in mathematics and argue that symbols get embodied and coded into various kinds of meaning thereby negating the arbitrariness that is the first impulse to symbolisation in logic and mathematics. This embedding of meaning into originally meaningless symbols as part of mathematical discourse, I suggest, is one possible way to distinguish between the use of symbols in logic and mathematics. In this sense, mathematics shows a tendency to move towards 'natural' signs that should in a sense remind us of the efforts of Indian logicians.

The nature of sign in Indian logic

Indian logic is an example of an approach to logic which has the idea of signs at its core. However, it is in some respects fundamentally different from Western logic.[1] Although the difference has often been stated to lie in the use of examples or in that of singular terms, I think it is clear that primarily there is a fundamental difference in their understanding of a sign, particularly a logical sign. Thus, the crux of the difference between these logical systems, in the context of semiotics, is between the notions of arbitrary and non-arbitrary signs.

Among the many classical Indian traditions, there is one important school called the Nyāya which is dedicated to logic. The earliest formulations of logic (primarily classification of different types of inference and structures of inference) are found in this tradition, including the well-known five-step process of logical inference.[2] A significant change in the analysis of logical arguments occurred with the Buddhists and in the succeeding centuries both these schools developed various themes in logic. Although it is often remarked that Indian philosophy is concerned with the mystical and the spiritual, it must be noted that concerns of logic and epistemology were central to most of the other traditions also.

Buddhist logic shows how logic is related to the nature of signs. A general theory of inference was developed in this tradition, which related logic and inference to the nature and role of signs. Subsequent developments in this field, largely arising from debates between the various Indian philosophical schools, critiqued and refined the many themes that occur in such an analysis of logic. One of the important insights in Indian logic has to do with the notion of a 'natural' sign, one that is sometimes synonymously used for reason and evidence. I introduce some basic ideas in Indian logic mainly to relate it to the notion of signs and their relation to logic, and also to understand the nature of arbitrary sign in contrast to necessary relationships between a sign and its signified.

A complete elaboration of the meaning of a sign, its relation to what it signifies, and the relationship between sign and inference had to wait for the great Buddhist logician, Dignāga (c. 450 AD). He turned the question of logic into a question of semiotics. Inference by its very nature is related to signs. Inference

[1] See Matilal 1999, Mohanty 1992 and Sarukkai 2005b.
[2] See Matilal 1999 and Sarukkai 2005b for an introduction to Indian logic.

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occurs when we come to believe in something which we don't directly perceive. Inference allows us to expand our perceptual capabilities, at least the 'perceptual' capabilities of the mind with the help of logical reason. Dignāga's logic is primarily an attempt to clarify what kinds of valid signs are possible and how we can make justified inferences from these signs. We can see an important difference between the early *nyāya* five-step process and Dignāga's formulation. In the *nyāya* process, the generalisation 'wherever there is smoke, there is fire' is one step in the five-step process. It is used as a part of the reasoning, whereas for Dignāga, it is precisely the reason which he wants to 'prove'. In this sense, Dignāga is concerned with justification of an inductive statement such as, 'Wherever there is smoke, there is fire'. He wants to know how an inductive cognition can be made certain.

There is yet another peculiarity in his formulation, and this has to do with the synonymic usage of sign, reason and evidence. Consider this commonly discussed example. From seeing smoke on a hill, we infer that there is fire on the hill. Smoke is the sign which indicates the presence of fire. Smoke is the evidence for believing that there is fire and smoke is also the reason for coming to the conclusion that there is fire. Thus, sign, reason and evidence are terms that are often used interchangeably in Indian logic. Dignāga's theory of inference sets out a structure of inference based on the nature of the sign, thereby defining when a sign can properly stand for another. He formulated the 'triple nature of the sign', three conditions which a sign must fulfil in order that it leads to valid inference.

1. It should be present in the case (object) under consideration.
2. It should be present in a *similar* case or a homologue.
3. It should not be present in any *dissimilar* case, any heterologue.[3]

The sign, as pointed above, is also the reason for the inference and is called the *hetu*. The inferred property is *sādhya* and location is *pakṣa*. For example, smoke is the sign for fire. To know if smoke is really a valid sign pointing to the presence of fire, we need to check whether it satisfies all the three conditions. Dignāga's first condition says that such a sign (smoke) should be present in the particular case under consideration and this is satisfied since smoke is seen on the hill. The hill is the subject-locus, the *pakṣa*. Seeing the smoke arising from the hill, we infer the existence of another property, fire. The second condition is that there should be examples of other similar locations which possess the sign (smoke). The example of the kitchen is one such, since the kitchen is also a place where we see smoke and fire together. We can understand this second condition as giving a positive example supporting or confirming the inference we make.

The third condition is a negative condition. It says that the sign, if it is to be a valid sign, must not be present in locations where the signified is not present. That is, smoke should not be present where fire is not present. If smoke were so present, then it would imply that there is no necessary connection between smoke and fire. In a sense, which will get clarified as we go along, Dignāga and

[3] Matilal 1999, p. 6.

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later logicians were trying to articulate what it means to have this 'necessary' connection although they did not phrase it in terms of necessity or, in general, in modal language. The example for the third condition is a lake which is not a locus of fire. So, if smoke is found over a lake where fire cannot, by necessity, be found then it surely rules out the necessary relation between fire and smoke.

To summarize the meaning of the above conditions: a sign which is present in a locus signifies another property of the locus. To have a degree of certainty about this signification, we need to find similar cases where the sign and the signified occur and also dissimilar cases as explained above. The occurrence of the sign and signified together is seen as illustrating a relation between them, the relation of invariable concomitance or pervasion.

It might be mistakenly thought that signs are restricted to material signs such as smoke. This is not true since sign and reason are used interchangeably. Thus, Dignāga's conditions are as much conditions which a reason should satisfy if it should be seen as being correctly associated with some thesis. Similarly for evidence. Suppose we say that we have evidence for some inference we make. Then to know if this is a valid evidence or not, we should check if the three conditions are satisfied. As we can easily see, this kind of check can be performed on any inference we make, including scientific inferences. Rewriting the three conditions in the following manner makes us see this character of the three conditions more clearly. These conditions can also be written in terms of the thesis (the inferential statement) and the reason (*hetu*) which is adduced as support for the thesis.

1. The *hetu* advanced in justification of a thesis must be relevant to the thesis.
2. It must support the thesis.
3. It must not support the opposite of the thesis.[4]

The fundamental issue raised by the approach of Indian logicians has to do with the relationship between a sign and the signified. In doing this, Indian logic seems to take a completely opposite view to that of Western logic. Both these systems, in general, understand the importance of signs in logical analysis.[5] However, in the development of mathematics, especially in the West, signs had already come to play a primary role. In particular, it was commonly felt that the fundamental advantage in the use of signs/symbols in mathematics (and in modern, symbolic logic) lay in the arbitrary nature of them, that is, in the arbitrary connection between the sign and signified. At one level, this suggests a profound difference between Indian and Western logic but at another it opens up new questions for mathematics and modern logic. In what follows, I will discuss in brief the trajectory of sign/symbol in Western thought and then conclude with a discussion of natural and arbitrary relations.

[4] Bharadwaja 1990, p. 11.

[5] There is yet another fundamental difference which I have characterised as follows: Indian logic is an attempt to make logic scientific, thereby turning on its head the question of what is prior - science or logic? For a detailed argument for this position, see Sarukkai 2005b.

The nature of symbols

The word sign is derived from *signum*, originally semeion, which was often a synonym of *tekmérion*, and was used to mean proof, clue and symptom.[6] These meanings share a common semantic space with the ideas of sign, reason and evidence, which are, as we have seen earlier, used in various ways in Indian logic. In the Western tradition, the Stoics had the 'first and most thorough sign theory every produced' and among the examples of inference, the smoke-fire inference was the 'most elementary type of recollectable sign.'[7] Eco notes that the Stoic model of sign is an inferential model of p implying q, 'where the variables are neither physical realities nor events, but the propositions that express the events. A column of smoke is not a sign unless the interpreter sees the event as the true antecedent of a hypothetical reasoning (*if* there is smoke...) which is related by inference (more or less necessary) to its consequent (... *then* there is fire).'[8] The sign is not the material sign of a particular column of smoke but is a type standing for smoke. In comparison to Indian logic, we might say that this is a sign removed twice, since the type smoke stands for a particular column of smoke which is the sign for fire (as in Indian logic). Furthermore, the inferences studied by the Stoics were not concerned with the epistemological relation between the terms in the inference, although Aristotle distinguished between necessary and weak signs based on epistemological concerns. Inference is a logical argument and the relation between signs and logic already appears here. However, there is a deeper engagement with logic and sign, one that needs to be explored before we understand the possible interpretations and extensions of Indian logic.

Signs play a fundamental part in human thought and they have a primary role in certain disciplines, particularly logic and mathematics. Frege believed that the idea of the sign was a 'great discovery'. He also held the position that ideas and concepts are possible only through the creation and use of signs. An added advantage of particular symbolic notations was that they did not manifest some common problems associated with verbal languages. However, note that sign here refers largely to an arbitrary set of symbols which are created by us with no specific *natural* meaning associated to them. In the same logical tradition, George Boole noted that signs are arbitrary as far as their form is concerned but once they have a particular interpretation then that should continue. He further added that the 'laws of signs are a visible expression of the formal laws of thought.'[9] Cassirer, much influenced by Leibniz, analysed in detail the central importance of symbols and science.[10] For him, the structure of science rests on the 'logic of things', namely, 'the material concepts and relations', and this logic of things cannot be separated from the logic of signs. An important consequence

[6] Eco 1984, p. 26.
[7] Ibid., pp. 213, 214.
[8] Ibid., p. 31.
[9] Boole 1997, p. 130 - 131.
[10] Cassirer 1953. See also Ferrari 2002.

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of this approach is his position that 'concepts of science are no more imitations of existing things, but only symbols ordering and connecting the reality in a functional way.' The importance of this view needs to be stressed again, especially the argument that concepts are also semiotic. This is important because it runs counter to the commonly held belief that the transition from observation to theory occurs through concepts and that concepts are neither linguistic nor part of a system of signs.

One of the most influential analysis of signs was given by Peirce, who also brought logic and semiotics together in an essential manner. Reflecting this, Peirce notes explicitly that 'Logic, in its general sense is ... only another name for *semiotic* (...), the quasi-necessary, or formal, doctrine of signs.' Peirce begins by defining sign, in a way similar to Dignāga, as 'something which stands to somebody or something in some respect or capacity.' There are three elements to a sign: the creation of another sign in the mind, the sign standing for an object, and the presence of an idea in reference to which the sign stands for the object. The second element of the sign, namely its capacity to stand for some other object, is what Peirce calls logic. Thus, 'logic proper is the formal science of the conditions of the truth of representations.'[11]

Peirce is a committed taxonomist. He classifies signs in great detail. Firstly, there are three types of sign for Peirce, what he calls three trichotomies. The first trichotomy consists of signs which is a 'mere quality, actual existent, or is a general law'. The second trichotomy includes signs based on their relation to the object and the third has 'sign of fact' or sign of reason. Each of these types has different kinds of signs within them. In the first trichotomy there are three kinds of signs: qualisign, sinsign and a legisign. A qualisign is a quality which functions as a sign to denote another thing - for example, the quality of red denoting a colour red. Our experience of red is not of the red colour or red thing in itself but is a sign pointing to the red object or red colour. A sinsign is an actual thing or event that occurs only once and acts as a sign. A legisign is a law which functions as a sign. Conventions are included under laws and Peirce gives the example of the usage of 'the' as having the same meaning as an example.

The better known classification of signs by Peirce are the three types of sign under the second trichotomy. These three types are called the icon, index and symbol. An icon is a sign that resembles an object, like a picture of a tree which has some semblance of similarity with the tree. An index is a sign that is a sign for an object which affects it and thus is modified by the object. A symbol is a sign that is accepted by convention as referring to an object.

Buchler in his introduction to Peirce's work notes that Peirce's path-breaking contribution is his 'conception of logic as the philosophy of communication, or theory of signs.'[12] He also says that the 'conception of logic as semiotic opens broad, new possibilities.' Arguably, we can well understand the aims of Indian

[11] Peirce 1955, p. 99.
[12] Ibid., p. xii.

logic alongside the approach towards logic and signs by Peirce. Peirce's ideas about signs share a common conceptual space with Dignāga and other Indian logicians. Therefore, it seems reasonable to claim that the ancient and medieval Indian logicians who based their logic on the nature of signs understood the essence of logic primarily as what in the Western tradition came to be called semiotics.

Duhem's conception of a symbol is an influential example of the importance of arbitrary symbols. Ihmig isolates five general features of a symbol for Duhem.[13] Firstly, signs are arbitrary and conventional. They do not possess any natural relation and therefore a sign does not have a natural connection with the signified. He also considers signs to be at a different level than phenomena. Thus, for Duhem, smoke cannot be a sign for fire since both smoke and fire belong to the same phenomenal level, whereas if smoke were to be a sign it has to be qualitatively different from fire. We can see that what Duhem posits is similar to the semiotic character of language, where a word is an arbitrary sign but it is also qualitatively different from the phenomenon/object it refers to. Further, for Duhem, signs are always part of a larger connected universe of symbols and therefore cannot be understood in isolation. The arbitrary nature of the symbol implies that there cannot be any truth-value associated with them. However, since a symbol also stands for something else, instead of the truth or falsity we can only ask whether it is appropriate or inappropriate. Finally, Duhem makes an important distinction between symbols used in scientific formulation and those that arise in ordinary generalisations. This point is similar to the ones made by many others about the difference between concepts in science and those in everyday life. Concepts in science undergo constant test, modification and rectification just as symbols in science, for Duhem, are similarly open to complex processes of creation and modification. Given Duhem's belief in the intrinsic relation between mathematics and theories, one can see that his formulation of symbols is very close to the mathematical use of symbols.

The emphasis on signs and symbols leading to their essential role in disciplines such as logic and mathematics is first initiated by the move of creating signs to stand for various elements of an observation. But, as our discussion on Indian logic clearly showed, this emphasis on sign and symbol is not special to the Western tradition. Given the insistence of Indian logicians to synonymously understand reason and sign, it is to be expected that the arbitrary nature of sign is not available in the logical formulations discussed earlier. But it would be wrong to say that the arbitrary nature of sign was not known to them, since Dignāga's formulation draws upon his *apoha* doctrine of language.[14] It was also very clear for Indian philosophers such as those belonging to the Nyāya tradition, that words function as signs standing for something else. Although the Grammarians and Mīmāmsakas subscribed to a naturalistic description of words and

[13]Ihmig 2002.

[14]See Siderits 1991 on the Buddhists views on language.

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meaning, the philosophers associated with the logical school, namely, Naiyāyikas and Buddhists, didn't do so. It was clear to these logicians that the relation between words and things was arbitrary. In fact, I think it is reasonable to argue that the stringent conditions on a valid sign may actually be a reflection of the problems of arbitrary connection between words and what they stand for. Since philosophy of language was one of the pillars of ancient Indian thought, the influence of these philosophies on logic might have succeeded in making the conditions on signs more rigorous than perhaps it otherwise would have been!

Therefore, if in the first place, the arbitrary relation of sign-signified is what is being sought to be eliminated in the three conditions of Dignāga, then it is no surprise that the kinds of valid signs in Indian logic are highly restricted. On the other hand, we can see that the use of symbolic notation in Western logic is itself a use of signs. Using a term to stand for something is actually to create a sign-signified relation. For the Indian logicians, this move is exactly the crux of the problem! What is it that allows us to use a symbol to stand for something else? How do we justify replacing the set of Greeks with the symbol A? In other words, what they are concerned about is the foundation upon which symbolism is possible, thus offering a challenge to the modern logicians to understand the presuppositions inherent in the very act of symbolisation.

So, once we use arbitrary symbols, we are not only inferring but also dealing with issues of language. If we are dealing with issues related to language, then the kinds of questions that arise are very different than if we are dealing with cognitive inferences. The use of arbitrary symbols for Indian logicians would be a movement into the domain of language, and thus perhaps be outside inferential cognition. This implies that even the attempts to symbolise Indian logic is to misunderstand something essential about it!

There is an interesting problem here. Indian logic arises in a culture which already possesses complex philosophies of language and thus responds to some of these issues. The question of arbitrariness of symbols is one such important issue. As we saw earlier, the relation between word and object can be natural or arbitrary. In the Indian systems, both these views are held. Since we are concerned with the logical tradition, we are justified in asking that since words as symbols are arbitrary and conventional, why is it that valid, logical signs are not so? Why wasn't there a semiotics of arbitrary symbols? Is it possible that the strong empirical content in these logical traditions came in the way of developing a semiotics of arbitrary symbols? Furthermore, is there a distinction between the arbitrariness of words and that of symbols?

Symbols, in some of the formulations discussed in the last section, are not based on the model of language, although they share the nature of arbitrariness with words. Rather, they are concerned with exhibiting two features: one, the distinction between the sign and what it stands for, and two, the possibility of what Leibniz called the 'universal characteristic' which will in some sense

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remove the arbitrariness in symbolic relations.[15] The arbitrary nature of the sign is only in its creation whereas for the Indian logicians the logical sign must have some necessary connection with the signified. Thus, there is a reason for our recognition of something as a valid sign. The question of validity of signs itself is quite special to Indian logicians, one which immediately negates arbitrary signs. The basic point is that while there can be a sign which can, in principle, stand for anything, the Indian logicians were concerned about finding the subset of these signs which have a special, natural relation with the signified. Since their logic was responsive to the concerns of language, arbitrary signs, for example, linguistic symbols, are already accepted into the system of signifiers. The synonymous use of sign, reason and evidence also points to the problem of viewing signs as being completely arbitrary.

Arbitrary symbols in certain Indian traditions such as Nyāya are exemplified by words. But the idea of a symbol as discussed by Leibniz and others is somewhat different from the arbitrary nature of words. Peirce also uses the example of words as symbols. However, the idea of an arbitrary symbol has an expanded interpretation, one which can be analysed by distinguishing between arbitrary symbols which can be both like words and not-words, implying thereby that there is a notion of symbol which differentiates between 'word-symbol' and 'non-word-symbol'. Arbitrary symbols can be classified in two ways with respect to meaning. Words, although arbitrary, are filled with meaning. Symbols, as used in logic and mathematics, are thought not to have an associated semantic world like the words. So, the relevant question that we need to consider is whether we can have symbols which are arbitrary but do not carry a space of meaning with them. It is important to note that the *meaninglessness* of symbols is important to make a transition into the logical and mathematical symbolical domain, and that this mode of arbitrary symbolisation is different from the arbitrary nature of linguistic words as symbols.

Therefore, the erasure of the originary question, namely, how signs get attached to their signifieds, is a question that must be first recollected in order to understand something essential about the nature of reasoning. Something similar is echoed in Goodman's new riddle of induction.[16] In the case of induction, Goodman suggests that how a habit gets formed is as important a question as to what justifies an inductive belief in the habit.

Symbols and Writing: The Example of Mathematics

What differentiates the use of symbols in logic and mathematics is the meaning that comes to be attached to symbols in mathematics. Symbols do not remain arbitrary and meaningless when taken into the fold of mathematical discourse whereas in logic there is still a dominant notion of arbitrariness that remains. I

[15] Ferrari 2002, p. 5.
[16] Goodman 1973. See also Sarukkai 2005b.

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also want to emphasise that an essential character of symbolisation lies in the process of writing itself. If we do not explicitly factor this character, then any discussion on symbols and symbolisation, especially as in formalism, remains incomplete.[17]

Mathematics is the best example of a discipline that essentially depends on the power of symbolisation. However, the notion of arbitrary symbols has been given undue importance in understanding the nature of this symbolisation. Applied mathematics, mathematics that is used in the sciences, poses a challenge to the arbitrary nature of symbols that occur in 'pure' mathematics. That meaning accrues to symbols is a possibility that mathematics has to accept. This is manifested in the practice of applied mathematics in many different ways. I will briefly discuss two uses of symbols in mathematics that demand a more sophisticated interpretation of signs in mathematics, one which includes the possibility of certain signs capturing a special relation to the signified.

The first is the process of what can be called 'alphabetisation' in mathematical discourse. The second is the process of meaning that accrues to symbols through the processes of 'pattern recognition' and the privileging of formal similarity in symbolic, graphic forms. One of the most important elements of symbolisation – one that is not often explicitly discussed – is the importance of writing to this process, one that is also related to the essentially written character of mathematics (and logic). Therefore, to understand the nature of symbols and their specific use in mathematics it will be useful to look at the writing of mathematics and in general the formation and structure of mathematical discourse.

The literature on the symbols of mathematics, such as constants and variables, is enormous. The attempt by Russell and Whitehead, in *Principia Mathematica* (PM), to rewrite mathematics entirely in the symbolic notation of logic is well known. Frege, Wittgenstein and others have contributed significant insights into the nature of the symbol in mathematics. But I shall not deal with their concerns here. Instead, I want to look at the use of symbols in mathematics along the trajectory of writing. My project here will be to exhibit the unique writing strategy of mathematical discourse that emphasizes the creation of new 'alphabets'. There are primarily two reasons for this unique creation of alphabets: they are immediately open to modes of calculation and the grapheme-like identity of these symbols comes to stand for Platonic entities.

Mathematics is a domain of symbols, what is usually called a semiotic system. But there is some ambiguity in the understanding of symbols. Black considers symbol as a 'word of the same logical type as *word*.'[18] This notion of symbols includes words and algebraic signs. Weyl considers four types of symbols, distinguished 'by the different rules of the game that apply to them.'[19] They are the constants, variables, operations and integrations. These symbols are different in

[17] For a detailed discussion of an analysis of mathematical discourse and the hermeneutics of symbolic use in mathematics, see Sarukkai 2002.

[18] Black 1965, p. 50.

[19] Weyl 1949, p. 55.

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character. Operators, for example, can be one-place, two-place and in general, many-place operations. In the PM there is a distinction made between complete and incomplete symbols. Operators fall under the latter category because they 'have no meaning in isolation and cannot be legitimately used without the addition of further symbols.'[20] Symbolic reduction, as in the representation of a proposition by a symbol p, is not the alphabetization that I refer to in this section. It is the *manipulation* based on the symbolic character, and not mere representation, that makes a symbol in mathematics an alphabet in the sense I describe below. It is this manipulation that infuses meaning into these symbols. In this context, it is worthwhile to note that in Wittgenstein's reaction against reducing mathematics to logic, similar arguments against the use of symbolizing statements in ordinary language to those of mathematical statements are offered.[21]

We commonly use alphabets such as x, y and n stand for numbers. This is the simplest example of creating alphabets by first choosing a few symbols and letting them refer to some mathematical object. Basically, alphabets of any language are possible candidates. This ability to absorb any graphic mark into the language of mathematics *as long as it is a grapheme* is already a fascinating move in the creation of this language. As long as it is *one continuous mark* on the sheet, it does not matter what it means or where it is derived from. Thus, alphabets in general become a part of mathematical language not because they are alphabets of natural language but because they are *individual, continuous marks.* Why is that numbers, for example, are denoted by single letters but not as a combination of letters – that is, why 'n' but not 'nu'? This is because mathematics constructs the mark of multiple letters (like xy standing for x multiplied by y) as standing for 'sentences' and more complex phrases. The attempt to 'create' alphabets is a discursive step and is the first step in writing the discourse that comes to be called mathematics.

The formation of alphabets and sentences is not merely shorthand in nature. They continuously enforce the agenda of the mathematical activity – calculation, proof and so on. This can be read along the trajectory of suspicion of natural language and verbosity that is inherent in the mathematical imagination. If mathematical essences are those that are extracted away from the verbosity of language, then the same inclination is also shown towards its very script. Words are not only ambiguous in the meanings associated with them. Even the extensionality of words, written as a combination of alphabets, is itself '*visibly verbose*'. Here is an almost innocent and nave picture of words as if they 'write' their verbosity in their 'length'! Should it surprise us that even in the English subtexts of a mathematical text rarely does one find a word that is 'long'?

We have to stick with this graphic reductiveness inherent in the creation of the mathematical language and look more carefully at the creation of the

[20]Black op. cit., p.76.
[21]See Marion 1998, chapter 2 and 6.

domain of alphabets in mathematics. Mathematics needs an alphabetic structure which, in principle, cannot conceivably be exhausted. The reason is obvious: as a discourse which claims only to expose a Platonic world, the perpetual possibility of finding 'new' objects, different in kind from the ones known, will necessitate the generation of new alphabets. We can now see why the reductive alphabetic mode is so important to mathematics – *alphabets refer to Platonic entities*. If this be the case, then mathematics has to have *discursive strategies to keep generating alphabets*. Here are a few examples of how this is done.

In the case of numbers it was simple. All that was needed was a letter. Now consider the example of a vector. Vectors specify both a number and a direction. A vector is commonly denoted by a letter (say v) with an arrow on top, where the symbolic presence of the alphabet v captures the numerical component and the arrow sign emphasizes the directional nature of vectors. It is also a common practice to print this in bold face as **v**, thus emphasizing the vector nature. Since these are conventions, we could well imagine that a vector could be represented by the use of two alphabets, one for number and another for direction. But mathematics is not written this way. It finds it preferable to draw an arrow on top of a letter or change its font or boldface it rather than represent it graphically with more than one letter. One may argue that it is the nature of symbolization that is behind this emphasis on grapheme-like structures. But symbols, essentially conventional, can also be created around more complex combinations of alphabets. In mathematics, the rewriting of the symbol, a grapheme yet not one, reflects the conscious attempt of the mathematical discourse to distinguish itself from natural language (NL). But in so doing, the alphabets no longer remain like the alphabets of NL; instead they begin to resemble 'pictures' as in the case of placing an arrow over a letter to denote a vector. More complex examples are considered below. This creation of pictures of symbols can be called the 'geometrization of words'. In an uncanny resemblance to the geometrization of the world, this move captures mathematical objects in the form of idealized single graphemes. As mentioned before, the identity of these symbols, as single, individual entities mimics the objecthood of the mathematical objects for which they take the place. The geometrization of words occurs in many ways and contributes to the originality that makes the semiotic system called mathematics so unique.

Consider the use of subscripts and superscripts. Consider Mersenne numbers: 'Numbers of the form $2^p - 1$, where p is prime, are now called *Mersenne numbers* and are denoted by M_p in honor of Mersenne, who studied them in 1644.'[22] The use of the subscript here retains the individuality of the number yet reminds us that it is unique, where the uniqueness is defined through its equality to $2^p - 1$. What this does, at the alphabetical level, is to create a set of potentially infinite objects, M_1, M_2 and so on. Classification of many kinds of groups also use subscripts for the purpose. The classification through the use of subscript, for example, embodies in the written form some characteristics of that mathematical

[22] Apostol 1976, p. 4.

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object. Subscripts are not only used to denote classes or families. In the case of vectors, the components of the vectors are usually written with subscripts and superscripts. Once these notations are used, vector multiplication, differentiation and so on, can get written in a manner that exhibits the process of operation in the very writing itself. For example, the scalar product of two vectors is written as $x_i y^i$ (this particular notation is a shorthand for the summation over the components). The 'cancellation' of the subscript and superscript suggests the scalar nature of the product on the order of *writing* the product. As any practitioner knows, the written forms of vectors and tensors (a tensor is usually written as $x^{abc...}$ or $x_{abc...}$ or $x^{ab..}{}_{c..}$) allow various manipulations of them, those that are suggested immediately by the written form of these entities.

The superscript has to be used carefully because an expression like 2^n is shorthand for operations, which in this case stands for 'two multiplied n times'. So the prevalence of classification by subscripts is more common than by superscripts but where such possible confusions may not arise, superscripts are also commonly used.

Yet another powerful method of geometrizing words is the use of brackets. Brackets play an important role in the project of alphabetization in mathematics. Primarily, it allows the use of more than one letter without reducing it to an expression of the natural language, like a word. One cannot continue to use single alphabets since the stock of these alphabets is soon exhausted. How can we now use expressions of more than one letter and yet use it to stand for one kind of entity? That is, have word-like structures that nevertheless continue to maintain an alphabetical character? There is also a related problem: in mathematics, two letters are usually used only when there is some operation involved. ab, which conventionally stands for a*b, where the * can be any operation, by itself does not refer to an object. It refers to a process whose end result, say c, will refer to a number, function, matrix or any other appropriate mathematical entity. In the symbolic notation a*b or ab is not a statement about an object but about a process.

Brackets allow the use of more than one alphabet and yet simulate the unity of an alphabet. A simple example is that of a function, written as f(x). This is not equivalent to 'fx'. f(x) is the geometrization which underlines the point that although f and x are alphabets of the English script, f(x) is not. The graphic mark of the brackets creates an enfolded singular 'grapheme', f(x). By the use of brackets, it creates the possibility of referring to a mathematical entity called the function, just as a, b 'refer' to numbers. The use of brackets is an important strategy of writing mathematics. They also occur in yet another important example, that of matrices. Matrices are arrays. They are represented in terms of numbers, functions etc. all placed along rows and columns. In the case of a 2 x 2 matrix, for example, if we remove the brackets we are left with four numbers 'hanging'. This pattern is meaningless and the use of the brackets brings these four elements together as one symbolic entity. The symbolic character is obvious because without any conventions of describing what matrices means,

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there is no way of adding or multiplying them. At the level of writing, what the brackets do is create the notion of a single, unified, continuous mark that will then refer to mathematical objects called matrices. Both f(x) and matrices show how brackets as graphical marks are used to create symbolic objects. We can easily extend this process of writing to more complex symbols. If the function depends on many variables then it is written as f(x,y,z...). In the case of matrices, the numbers inside the brackets can themselves be replaced with other matrices, or functions or whatever! *It is the 'writing' of the bracket that gives the identity of the matrix to itself.*

These examples powerfully illustrate that mathematics has to be essentially written. One cannot read subscript and superscript (unless it is like 2^n which is read as two to the power of n). How do we 'speak' F^n as F^n and not as Fn? We can recollect Derrida's play on difference and *differance* that works on the indistinguishability in speech but not in writing. Is the mathematical language similarly oriented? F^n, F_n, F(n) and Fn are all indistinguishable in speech but are written differently. One might respond here that we read F_n as 'F subscript n' or F(n) as 'function of n'. But to do this is to put language back into the symbols, to put in the extensionality of words whose removal, in the first instance, allowed the formation of the language of mathematics. By explicitly reading the language lost in the reduction, we do not read mathematics but something else. It is also curious that the brackets, which play such an important role in creating the basic mathematical symbols, are verbally silent! We cannot 'speak' the bracket. Would we read F(n) as F open bracket n close bracket?

Alphabets of NL are fixed. English has twenty-six of them and exactly that. We do not create new alphabets, we create words instead. This creation is linear and not geometrical. We do not put arrows on the word vector to indicate its directionality. We 'speak/write' its properties. We do not put subscripts and superscripts to words. (And where we use brackets in natural language texts, it is to say things that we do not want to say as 'part' of the text.) In sticking with the fixed alphabetical system of NL, we have little choice open to us but even this restricted choice has created millions of texts! Because we are doomed (why?) to create words linearly, with one alphabet following another, the discourses based entirely on NL reflect the consequences of this convention. I have argued that mathematical language creates its alphabets in the ways indicated above. Once it does this, it can continue to write its narrative. *To write using these alphabets is to calculate with them and through them.* This is what is meant by saying calculation is the name for writing in mathematics. In this writing, the first step is the creation of arbitrary symbols, which soon after being absorbed into mathematical discourse loses this sense of arbitrariness.

The creative use of symbols in mathematics in the ways described above indicates how the use of symbols in mathematics differs from that in logic. An important consequence of this unique nature of symbolization in mathematics is the ability to read meaning from formal terms. The capacity to identify formal terms as having the same meaning is of profound importance in applied

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mathematics and is a discursive mechanism that helps to make mathematics 'unreasonably effective'. It is worthwhile recollecting here Frege's observation that it is the applicability of mathematics that makes it more than a game.[23] This also implies that arbitrary symbols derive meaning through the process of applicability and in so doing, lose their arbitrary symbolic status. Now we can see another way of understanding the difference in the use of symbols in mathematics and logic. Symbols derive meaning in mathematics both because of the Platonic impulse (as discussed earlier) and also because of its applicability. Logic does not exhibit similar concerns. These distinctions also succeed in distinguishing the role of symbols in both these disciplines thereby suggesting that logicism misses the point in trying to place mathematics within logic. In the following section, I will argue how formal similarity functions best at the level of symbols and how this suggests a picture view of symbols. This analysis will bring us back to the notion of necessary connection between a sign and its signified, in a manner not unlike that of the Indian logicians.

Symbols, written form and similarity

The idea of similarity is fundamental to the *writing* of the discourse of mathematics and science.[24] For example, many problems in physics are classified into certain classes. The set of problems that fall under the class of harmonic oscillators is an illustrative example. Various physical situations such as the dynamics of pendulum movement, springs, and in general the motion of particles near a position of equilibrium can be modeled in this class of harmonic oscillators.[25] There is a formal similarity based on graphs, pictures, diagrammatic representation and written form that makes this possible. Here, I will limit myself to the presence of written form alone since the arguments for the other graphic representations are very similar.

In classical physics, the problem of two particles interacting with each other is an important one. This problem is used to model various physical situations. Let me consider one particular approach to this problem. I will follow the classical text on mechanics by Landau and Lifshitz.[26] Let us denote by m1 and m2 the two masses of the particles, and v_1 and v_2 their respective velocities. The kinetic energy of each particle is $\frac{1}{2}$ $m_1 v_1{}^2$ and $\frac{1}{2}$ $m_2 v_2{}^2$. The total energy of the system is the sum of these kinetic energies plus the interactive potential energy term. The physics of the system allows us to write the sum of the kinetic energies of the two particles as one term, $\frac{1}{2}$ mv^2, where m is related to the two masses by the equation $m = m_1 m_2/(m_1 + m_2)$, and v is related to the two velocities through

[23] See Dummet 1994; Also Sarukkai 2003 and 2005a.
[24] For more details on this topic see Sarukkai 2002.
[25] See Kibble and Berkshire 1985.
[26] Landau and Lifshitz 1976.

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the equations $v_1 = m_2v/(m_1 + m_2)$ and $v_2 = - m_1v/(m_1 + m_2)$.[27] m is called the *reduced mass.* The form of the kinetic energy after the *rewriting* immediately suggests a *formal resemblance* of this term with the kinetic energy terms for each of the two particles. Because of this similarity we are led to consider whether the two particle system could indeed be 'seen' as the motion of one particle that has a mass 'm' and velocity 'v'. It is not surprising that the authors point to this similarity and refer to it as being 'formally identical.'[28] Thus, they conclude, 'the problem of the motion of two interacting particles is equivalent to that of the motion of one particle in a given external field U(r).'[29]

It is clear that a specific notion of similarity is alluded to in this case. Obviously, the two-particle motion is neither 'identical' nor 'similar' to the motion of one particle, at the phenomenological level. It is only the *expression* of the two-particle motion that seems to be 'equivalent' to that of one particle motion, under certain conditions.[30] This equivalence is suggested *only* because there is a formal similarity in the expressions of kinetic energy, in the way it is *symbolically written.*

This strategy is manifested at the most fundamental level. It is important to note that not only mass, but also velocity and kinetic energy, have now become forms that can be used as comparisons to a new content that is generated. Kinetic energy as $\frac{1}{2}$ mv^2 is the simplest expression of it. As new theories develop, they do not jettison the idea of kinetic energy, even though fundamental worldviews may be discarded. Rather this concept is placed along the trajectory of other related concepts like energy, potential energy, momentum and so on. It is also held onto *as a form* in field theories and quantum physics, even though the expressions for it have continuously changed. For example, consider the wave equation. The 'kinetic energy' term for a wave, $\frac{1}{2}$ $(\theta\mu\Phi)^2$, is 'formally identical' with that of $\frac{1}{2}$ mv^2. The velocity term here is the derivative of the wave function Φ, and is expressive of an entirely different content. But the similarity of the form immediately suggests its connection with kinetic energy. Once again, although the 'meanings' are radically different, the similarity is based on identification of forms. This is indeed very common in physics and not restricted to the example of kinetic energy. When new expressions are generated, formal similarity plays an important role in identifying and assigning it to prior named terms. What allows the retention of the name is similarity to certain forms, either as $\frac{1}{2}$ mv^2 or being related to momentum as p^2/2m where 'p' stands for the momentum, or through Virial Theorem and so on.[31] In this context, it should not be surprising to note that Landau and Lifshitz refer to the kinetic and potential energies as *names.*[32]

Such a process takes place even across paradigmatically different theories.

[27]Ibid., p. 29.
[28]Ibid., p. 29.
[29]Ibid., p. 29.
[30]We should remember here that there are assumptions in this reduction, notably that of isotropy of space.
[31]The virial theorem relates the time average values of the kinetic and potential energies. See Landau and Lifshitz, ibid., p. 23.
[32]Ibid., p. 8.

The Schrodinger equation is the most fundamental equation in quantum mechanics.[33] The *form* of this equation is exactly the same as the equation for classical particles, although this equation is a wave equation! The 'kinetic energy' term in Schrodinger's equation is also exactly identical – formally – to the kinetic energy of the particle written in the form $p^2/2m$, although there is a fundamental and paradigmatic shift of understanding momentum and energy as 'operators' rather than values.

Steiner offers similar arguments in his book.[34] He aims to show that 'Pythagorean' and 'formal' analogies are constantly used in the creation of new physics. He considers Pythagorean analogies as being entirely mathematical in contrast to physical analogies. By formalist analogy, he means 'one based on the syntax or even orthography of the *language* or *notation* of physical theories.'[35] Earlier on in the book, he writes:

> 'In some remarkable instances, mathematical *notation* (rather than structures) provided the analogies used in physical discovery. This is particularly clear in cases where the notation was being used without any available interpretation. So the analogy was to the form of an equation, not to its mathematical meaning. This is a special case of Pythagorean analogies which I will call formalist analogies.'[36]

He also discusses the example of the Schrodinger equation. There are two formal analogies in this case. He points out that this equation is '*formally* identical to the equation for a *monochromatic* light wave in a *nonhomogeneous* medium.'[37] He also informs us that 'Schrodinger himself tells us that his relativistic equation is based on a purely formal analogy.'[38] Steiner considers many other examples in physics that use analogies of the above kinds to prop up his basic thesis that a naturalistic account of the activity of physics is not possible; rather, this activity is essentially an anthropocentric one.

Although his arguments share a common space with mine, primarily on the emphasis given to the formal similarity of the inscriptions, there is a divergence on what we want to do with the identification of such similarities. Steiner consistently uses these examples to argue against naturalism; I use it to deliberate on the ideas of similarity and comparison that the use of such formal expressions entails. In particular, I suggest that one of the ways in which science captures the form of the world is to first capture it in its written form. Similarly, mathematics first captures the form of a Platonic world as written forms. As a consequence, the fundamental privilege given to the notion of similarity is manifested in the most important criteria for similarity: that between the world and the discourse of science or that between the Platonic world and mathematics.[39]

[33] See, for example, Landau and Lifshitz 1977, p. 50.
[34] Steiner 1998.
[35] Ibid., p. 54.
[36] Ibid., p. 4.
[37] Ibid., p. 79.
[38] Ibid., p. 99.
[39] For more on this, see Sarukkai 2002.

We can now see the potential problem that an analysis of symbolization in mathematics creates for the idea of arbitrary symbols. Consider the example of kinetic energy again. Assuming 'm' and 'v' to represent mass and velocity of an object, we can construct a new symbol, mv^2. Now, if the earlier symbols were arbitrary, then it should follow that any complex symbol formed from the simpler ones must also be arbitrary. However, the term $\frac{1}{2}mv^2$ is the 'sign' for a property of the moving object, its kinetic energy. Interestingly, this sign for kinetic energy is the valid and necessary sign for it; no other combination of the simpler symbols can stand for kinetic energy. Thus, from a combination of arbitrary symbols we seem to have ended up with a sign which is in a necessary relation with the signified.

This shift from the arbitrary to the necessary is not restricted to the example of kinetic energy but is found in all physical concepts that have a mathematical sign representing them. In fact, this natural association is a very important methodological tool for theoretical research, since it allows us to detect where physical concepts could lie hidden in some mathematical description. Earlier in this paper, I had suggested that arbitrary symbols can be word-like and not-word-like. The difference between these types of arbitrary symbols is that the word-like ones are arbitrary but also have meaning associated with them or generated around them through use in various contexts. The non-word-like symbols do not have a semantic space associated with them. $\frac{1}{2}mv^2$ is not only not arbitrary but it now has meaning, which is captured by its formal structure. Although we can still replace this symbol by any arbitrary, meaningless symbol like k, the meaning associated with it obstructs this move.

Although the above conclusions seem to follow from 'applying' mathematics, such as in physics, it is nevertheless the case that symbols with and without meaning occur both in pure and applied mathematics. One of the reasons why mathematics is so effective in describing the real world lies in its ability to match the signs in its discourse with some physical concepts, like the relation between kinetic energy and $\frac{1}{2}mv^2$. It is clear that there is no causal link between these signs and the appropriate physical concepts. But then what gives the notion of necessity in this relation? In Buddhist logic, inference is classified broadly into two types: as being of own-nature and as in causal relation.[40] It has been suggested that own-nature inferences are like analytical statements.[41] Associating specific complex signs for specific physical concepts is made possible through definition. Thus, we define kinetic energy to be $\frac{1}{2}mv^2$ but this is not an arbitrary definition. We cannot define kinetic energy in any other way. And in representing kinetic energy mathematically, we can only use this sign or its symbolic equivalents (like $p^2/2m$, where p is the momentum, which classically is equal to mv).

This suggests that the analysis of the sign-signified relation in the use of mathematical symbols for physical concepts falls under a particular analysis of

[40] This is the classification given by Dignāgas successor, Dharmakīrti. For more on Dharmakīrtis three-fold classification, see Sarukkai 2005b.

[41] See Matilal 1999.

signs. And this analysis is well described by Dignāga's basic question: When is a sign a logical sign? To paraphrase it for the example discussed above, we can ask when is the sign for kinetic energy one which really stands for the physical concept 'kinetic energy'? Although not so obvious, there are also cases of symbolic use in 'pure' mathematics where meaning accrues to symbols is similar ways; the process of alphabetisation discussed earlier is one such process. More detailed analysis of this issue will take me too far away from what I want to do in this paper but already in this approach we can see the potential use of drawing upon a different system of logic, namely, Indian logic, to explore new ideas regarding the process of symbolisation.[42]

It is clear that symbolization does play an important role by creating the possibility of formal similarity *at the level of language*. This is an important issue in the formation of the discourses of mathematics, science and logic, and is also related to the suspicion of natural language inherent in these discourses. The fundamental problem here is that the idea of similarity in natural languages is extremely difficult to grasp. Natural language, in the graphic mode, does not also allow for the possibility of similarity of forms. Two words have no relation of similarity although their figural inscriptions may seem to show it. Two words 'close' to each other in the written mode can be totally dissimilar in their meanings. The example of 'word' and 'world' is already a powerful one. Similarity, in the context of writing natural language, can arise only in the context of meaning, reference, actions associated with the word and so on.

But for mathematics (as well as for science, particularly in its use of mathematics), it is very important to manifest and use similarity at the level of language itself. This attempt is well captured in the symbolic mode and the shift to mathematical writing. Mathematics, in this context, functions as the linguistic form, the *figure of language* itself. Two mathematical expressions, which graphically look alike, are indeed 'close' and 'similar' to each other, unlike the case of words in natural language. More complex 'sentences' of mathematics also follow this rule. New expressions that are formed and written have to enforce the similarity and regulate the content by the boundary presence of the *form of graphic inscription*. In symbolic and mathematical language, the content is already in the form, thereby negating excessive preoccupation with the question of meaning behind and beyond the graphical writing. Is it any wonder then that mathematics and science, in general, rarely address the question of meaning in their discourses since the idea of similarity is already grasped in their writing? In other words, if what we say (and what we write) is exactly what we see, then the idea of the simulacrum has to be co-opted not only as a response to image, but also to *writing as image*.[43]

More importantly, these writing strategies based on formal similarity in the written form, as also in the figures, graphs, etc., generate knowledge. New struc-

[42] For a discussion on related issues, especially in the context of applicability of mathematics, see Sarukkai 2003 and Sarukkai 2005a.

[43] For a discussion on writing, simulacrum and image, see Sarukkai 2002.

tures, insights and theories are made possible, not by experiments or by logical arguments alone, but also by following certain strategies of writing related to symbols. This is yet another mark of distinction between logic and mathematics, one that is exhibited at the level of symbols and symbolic writing.

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